Implications for $d=4$ Ising model

In $d=4$, u is marginally irrelevant and leads to logarithmic corrections over the Gaussian result. The RG eqns we derived are:

$$\frac{dt}{dl} = 2t + 12u(\Lambda^2 - t)K_4$$

$$\frac{du}{dl} = -36u^2 K_4$$

They have an unseemly property that the cutoff Λ appears explicitly. One can actually drop the term involving Λ without changing any universal physics (exponents etc.) – this merely has the effect that the fixed point value of t^* is now zero. The gain is that the RG eqns are simplified. Further, we rescale $uK_4 \longrightarrow 2u$ (this convention leads to RG eqns that are identical to those obtained via the method of operator product expansion, which we will discuss later).

Thus, the RG eqns become,

$$\frac{dt}{dl} = 2t - 24ut$$

$$\frac{du}{dl} = -72u^2 .$$

We consider RG flow between two points; $(t(l=0), u(l=0)) \equiv (t, u(0))$ and $t(l_0), u(l_0)$. Here l_0 is chosen so that one is sufficiently far from the critical point i.e. $t(l_0) = O(1)$. Therefore, one can trust mean-field theory at l_0. In contrast at $l=0$, one is in the critical regime. Our aim is to find physical quantities such as the free energy, specific heat etc. at $l=0$ as a f^n of t.

Integrating the second eqn, one finds

$$u(l) = \frac{u(l=0)}{1+72ul} = \frac{u(0)}{1+72u(0)l}$$

Clearly as $l \gg 1$, u decreases as expected for a marginally irrelevant variable.

Substituting the expression for $u(l)$ in the eqn for t and integrating.

$$\int_{t}^{t(l_0)} \frac{dt'}{t'} = \int_0^{l_0} \left[2 - \frac{24\, u(0)}{1+72\, u(0)\, l}\right] dl$$

$\Rightarrow$

$$\log\left(\frac{t(l_0)}{t}\right) = 2 l_0 - \frac{1}{3} \log\left[1 + 72\, u(0)\, l_0\right]$$

From this, one may approximately solve for l_0 as.

$$l_0 \sim \frac{1}{2} \log\left(\frac{t(l_0)}{t}\right) + \frac{1}{6} \log\left[1 + \frac{72\, u(0)}{2} \log\left(\frac{t(l_0)}{t}\right)\right] + \cdots$$

Now, we can calculate the free energy.

Recall, $f(t, u(0)) = b^{-d} f_s(t\, b^{y_t}, u\, b^{y_u})$

Here $b = e^{l_0}$, $t\, b^{y_t} = t(l_0)$, $u\, b^{y_u} = u(l_0)$.

Since at $l = l_0$ mean-field theory works,

$$f_s(t(l_0), u(l_0)) \sim \frac{t^2(l_0)}{u(l_0)}$$

$$\Rightarrow f_s(t, u(0)) \sim e^{-4l_0} \frac{t^2(l_0)}{u(l_0)} \qquad (d=4).$$

Using the above expression for l_0,

$$e^{-4l_0} \sim \frac{t^2}{t^2(l_0)} \log \left[1 + 36\, u(0) \log \frac{t(l_0)}{t}\right]^{-2/3}$$

$$\frac{t^2(l_0)}{u(l_0)} = \frac{t^2(l_0)}{u(0)} \left[1 + 72\, u(0)\, l_0\right]$$

$$= \frac{t^2(l_0)}{u(0)} \left[1 + \frac{72}{2} u(0) \log \frac{t(l_0)}{t} + \cdots \right]$$

$$\simeq \frac{t^2(l_0)}{u(0)} \left[1 + 36\, u(0) \log \frac{t(l_0)}{t}\right]$$

$$\Rightarrow f_s(t, u(0)) \sim t^2 \left[1 + 36\, u(0) \log \frac{t(l_0)}{t}\right]^{1/3}$$

Therefore, compared to the mean-field result $f_s \sim t^2$, one obtains a log correction of the form $(\log t)^{1/3}$.

Specific heat $c \sim \frac{\partial^2 f_s}{\partial t^2} \sim \left|\log(t/t_0)\right|^{1/3}$ at the leading order.

Generalization to O(n) model and SAW exponent:

Similar calculation as above leads to the following RG flow for the O(n) model at $O(\varepsilon)$:

$$\frac{dt}{dl} = 2t - 8(n+2)ut$$

$$\frac{du}{dl} = \varepsilon u - 8(n+8)u^2.$$

where we have again dropped the cut-off dependent term in $\frac{dt}{dl}$. The derivation is left as a homework. From this, one obtains,

$$y_t = \nu^{-1} = 2 - \frac{n+2}{n+8}\varepsilon$$

As discussed earlier, the self-avoiding walk can be mapped to $n \to 0$ limit.

$$\Rightarrow \quad \nu^{-1}_{SAW} = 2 - \frac{\varepsilon}{4}$$

$$\Rightarrow \quad \nu^{-1}_{SAW} = \frac{1}{2} + \frac{\varepsilon}{16}.$$

Setting $\varepsilon = 1$, one obtains $\nu_{SAW} \simeq 0.5625$ which is not too far from the known result $\nu_{SAW}(d=3) \simeq 0.58$. Again, this can be systematically improved upon. For example, at $O(\varepsilon^2)$, using the expression in Wilson-Kogut article, one finds, $\nu_{SAW} \simeq \frac{1}{2} + \frac{1}{16} + \frac{120}{8^4} \simeq 0.59$.